LD's MATHEMATICAL SERIES

GRAPHS

ND IMAGINARIES

HAMILTON & KETTLE

1/6

GRAPHS AND IMAGINARIES

GRAPHS AND IMAGINARIES

AN EASY METHOD OF FINDING GRAPHICALLY IMAGINARY
ROOTS OF QUADRATIC EQUATIONS AND IMAGINARY
POINTS OF INTERSECTIONS OF VARIOUS CURVES,
WITH ILLUSTRATIONS OF THE PRINCIPLE
FROM ELEMENTARY GEOMETRY

BY

J. G. HAMILTON, B.A.

AND

F. KETTLE, B.A.

AUTHORS OF "A FIRST GEOMETRY BOOK."

PREFACE.

IN the following pages we develop a method by which imaginary roots of a quadratic equation can be found graphically as easily and as accurately as real roots, and we extend it to the location of other 'imaginary' points. The principle is clearly capable of much wider application.

As the parabola enables us to find the roots, when real, of any quadratic equation by suitable movement of the axes there would seem some warrant for the conclusion that an intimately related curve would perform a similar function for imaginary roots and thus make the graphical representation complete. Experiment and investigation confirm this inference, and a few typical cases chosen from Elementary and from Analytical Geometry are here worked out in detail in illustration of the principle.

<div style="text-align: right">

J. G. H.

F. K.

</div>

CLAPHAM, S.W.
        Jan. 1904.

CONTENTS.

GRAPHS AND IMAGINARIES

Roots of Quadratic Equations from Parabola graph.

Fig. 1 gives the graphs of x^2 and of $-x^2$ and, since the abscissa of any point on the curve is the square root of the ordinate of that point, it is immediately evident from their construction that

$$\sqrt{OK} = \pm KM$$

and $$\sqrt{-OK} = \pm i \cdot KM,$$

kO being equal to OK and km to KM.

In other words, if we move the x-axis up or down, the intercepts between the axis of the parabola and the curve itself give us the roots of the equation $x^2 = n$, for all values of n, positive or negative. Thus we at once read off the approximate roots of

$$x^2 = +3, \text{ as } \pm 1\cdot 73,$$

and of $$x^2 = -3, \text{ as } \pm 1\cdot 73\, i,$$

the i being omitted or inserted just as the roots are read off from the real curve (x^2), or its shadow $(-x^2)$ shown by the dotted line.

Now since we can express any quadratic equation

$$A x^2 + Bx + C = 0,$$

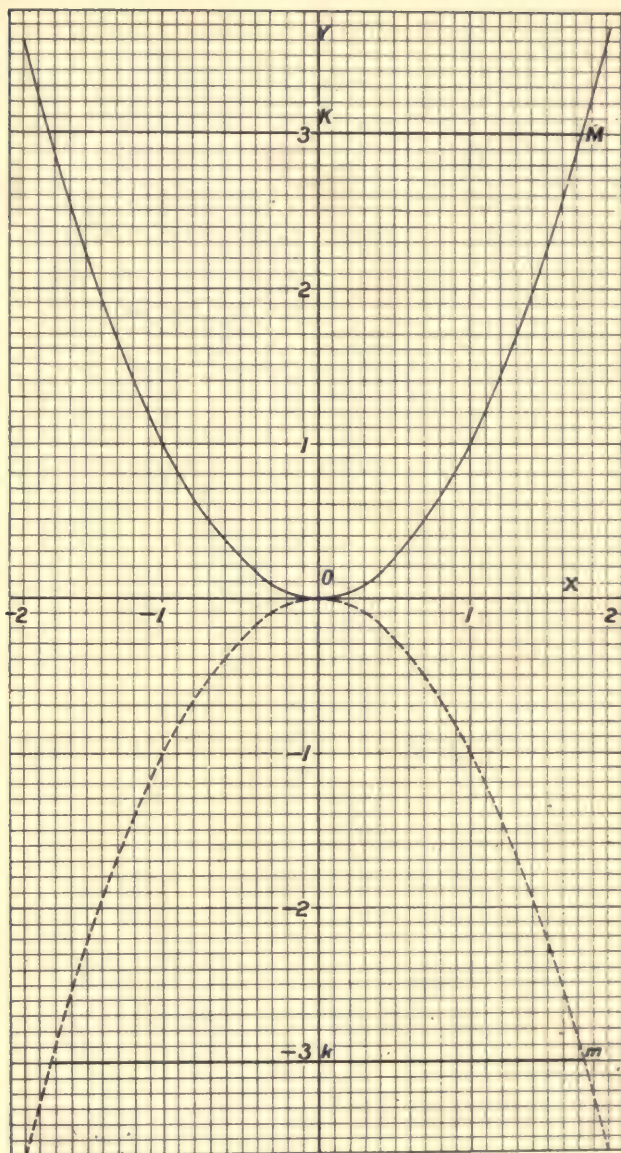in the form $(x + a)^2 = n$, and since also if we have the graph of

Fig. 1.

x^2 with respect to a pair of axes OX, OY (Fig. 2), we at once get the graph of $(x + a)^2$ by simply moving the y-axis through a distance of a, to the right if a is positive and to left if a is negative, it follows that if we can read off imaginary roots of $x^2 = n$, we can do the same for $(x + a)^2 = n$, i.e. of course for the general equation $Ax^2 + Bx + C = 0$.

The roots of the equation $(x + a)^2 = n$, will be

$$O_1O \pm KM \text{ if } n \text{ is positive,}$$

$$O_1O \pm i \,.\, KM \text{ if } n \text{ is negative (Fig. 2),}$$

where $O_1O = -a$, and $KM = km = n$.

This is illustrated in Fig. 2 for the case in which $a = -1$, $n = \pm 3$. If OX, OY were the axes the continuous curve would be that of x^2; push the y-axis 1 to the left and regard O_1X, O_1Y_1 as the axes and the continuous curve is now the graph of $(x - 1)^2$ with respect to the new axes. In the figure $O_1O = 1$, $OK = +3$, $Ok = -3$, hence we read off the roots of

$(x - 1)^2 = +3$, or $x^2 - 2x - 2 = 0$, as $1 \pm \sqrt{3}$ or $1 \pm 1\cdot73$

and of

$(x - 1)^2 = -3$, or $x^2 - 2x + 4 = 0$, as $1 \pm i\,.\sqrt{3}$, or $1 \pm 1\cdot73i$

approx.

In other words, to find by help of a parabola the roots of an equation given in the form $x^2 - 2x + 4 = 0$, we may plot the graph in the usual way. If we did so, taking O_2X_b and O_2Y_1 as the axes of x and y, the graph would come in the position shown by the continuous curve in Fig. 2. As this does not cut the x-axis the roots are clearly imaginary, but these can be read off almost as readily as real roots from the intersections of the shadow curve with the x-axis, the roots being $O_2k \pm i\,.\,km$. But of course there is no necessity to draw the shadow parabola at all, for we may simply imagine
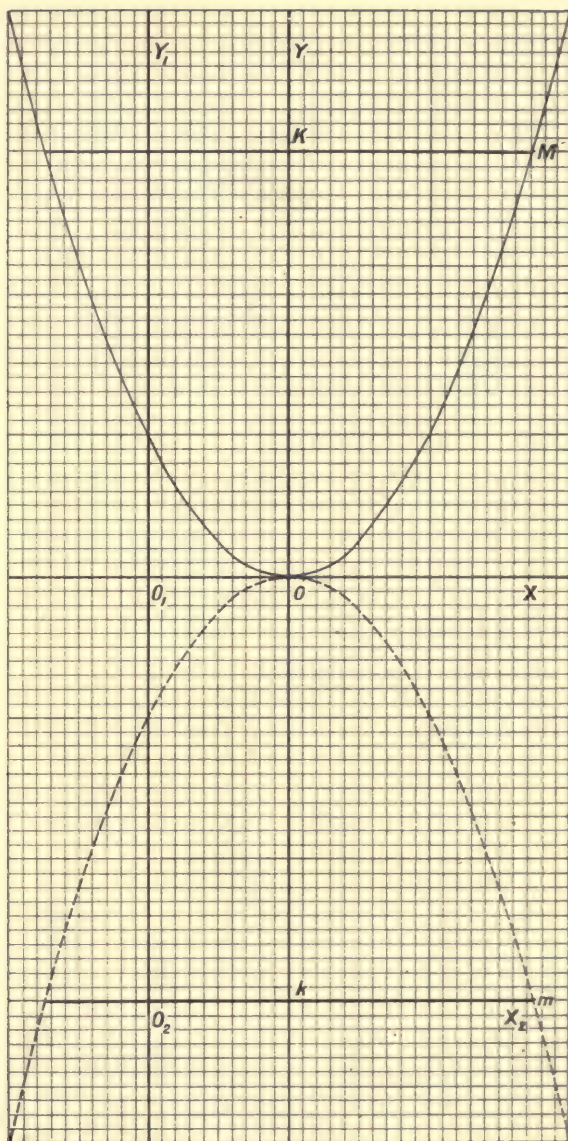
FIG. 2.

the x-axis pushed up through twice the minimum value of the corresponding function*. This latter method is in agreement with the fact that if we take any quadratic equation,

$$ax^2 + bx + c = 0 \dots\dots\dots\dots\dots\dots(1),$$

and form a new equation

$$ax^2 + bx + c = 2m \quad \dots\dots\dots\dots\dots(2),$$

m being the minimum value of $ax^2 + bx + c$, the roots of the derived equation are the same as those of the original except that the discriminant of the derived is i times the discriminant of the original equation. Reverting to the case illustrated by Fig. 2, since twice the minimum value of

$$x^2 - 2x - 2, \text{ i.e. } (x-1)^2 - 3 \text{ is } 2(-3)$$

we have, for original and derived equations,

$$x^2 - 2x - 2 = 0 \quad \dots\dots\dots\dots\dots\dots(1),$$
$$x^2 - 2x - 2 = 2(-3) \quad \dots\dots\dots\dots \dots(2),$$

the roots being $1 \pm \sqrt{3}$, and $1 \pm i\sqrt{3}$ respectively.

Now since the graphic solution of a quadratic equation by the above method resolves itself into finding the points of intersection of a parabola with the axis of x (or with some line parallel to it) we are led on to consider the case in which the straight line has some other direction, and that case in particular in which the straight line lies entirely outside the parabola, the points of intersection being therefore imaginary. The equations

$$y = x - 2, \quad y = x^2 - 4x + 5$$

may be taken in illustration. Their graphs, MVN and mn, are given in Fig. 3, the *shadow* parabola (dotted) is also shown, and also the given line mn, first pushed parallel to

---

* The lowest point of the curve being of course graphically determined, if necessary, by the perpendicular bisector of any chord of the parabola parallel to the x-axis.
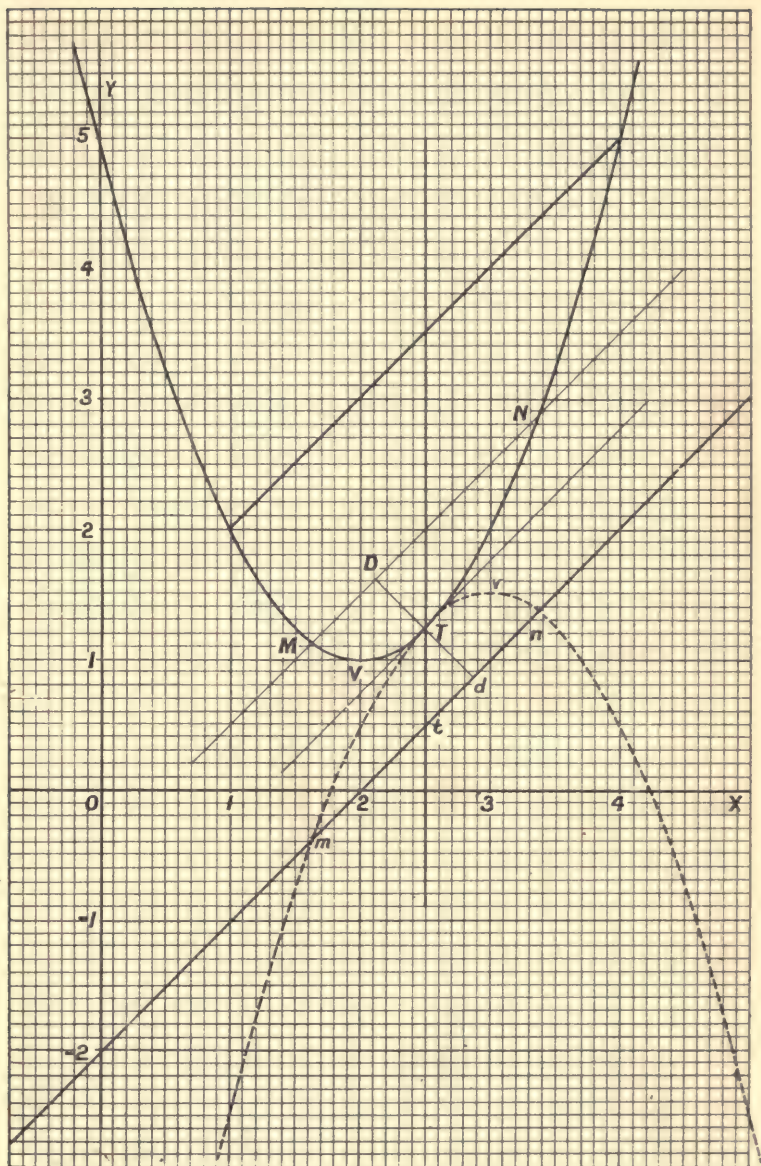
FIG. 3.

itself to a tangential position at T^*, and then further on for an equal distance (so. that $TD = dT$), cutting the real curve at the points M and N.

The roots of the equations may be obtained either by help of a shadow parabola which can be obtained from the original by revolution about the tangential point T (and easily shown with tracing paper), or by projecting the points M, N and T on to the given line mn, the projectors being parallel to the y-axis. The readings are :—

For x. Abscissa of $t \pm i$ (difference of abscissae of n and t).

For y. Ordinate of $t \pm i$ (,,   ,, ordinates   ,,   ,,).

In the particular case illustrated, the readings on inch and tenth-inch paper will be approximately,

$$x = 2\cdot5 \pm \cdot87i, \quad y = \cdot5 \pm \cdot87i.$$

By calculation

$$x = 2\cdot5 \pm \cdot5\sqrt{3}\,i, \ \ y = \cdot5 \pm \cdot5\sqrt{3}\,i.$$

The tangent parallel to the line $y = x - 2$, touches at the point $2\cdot5$, $1\cdot25$.

---

* T being found practically by the bisecting of any chord of the parabola parallel to mn by a line parallel to the axis of the parabola.

Circle method of finding roots of a quadratic equation.

The same principle is readily adapted to what, for brevity, may be called the circle* method of solving quadratic equations. This is illustrated in Fig. 4 for the equation

$$x^2 - 3x + (\tfrac{1}{2})^2 = 0,$$

where $AB = 3$, the sum of the roots,

and $AR \cdot BS = (\tfrac{1}{2})(\tfrac{1}{2})$, the product of the roots,

and in Fig. 5 for the equation $x^2 - 3x + (\tfrac{5}{2})^2 = 0$, AB again representing the sum, and $AR \cdot BS$ the product $(\tfrac{5}{2} \cdot \tfrac{5}{2})$ of the roots.
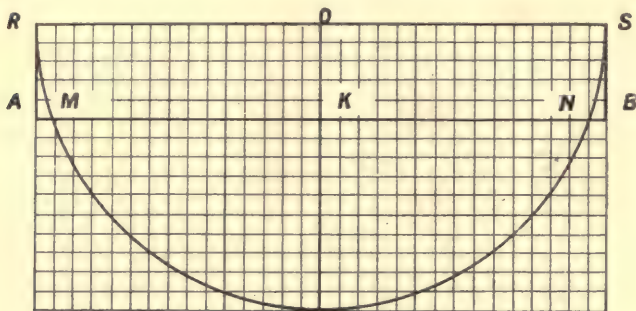

Fig. 4.

* This method of obtaining a first approximation is treated of in H. Bos, *Éléments de Géométrie*, p. 162 (3rd edition, 1888); Chrystal, *Introduction to Algebra*, p. 332; Lodge, *Differential Calculus for Beginners*, p. 91, &c.; Mackay's *Euclid*, vi, 28, 29.

If the circle cuts AB in M and N we have *real* roots which can be read $AK \pm KN$; if it does *not* cut AB the roots are *imaginary*. Taking $x^2 - ax + b = 0$ as our standard equation (a and b positive), keeping a constant, and giving b continually increasing values, we shall find

(1) that the circle gets further and further away from AB,

(2) that the roots are

    $AK \pm$ a continually decreasing quantity,

then $AK \pm 0$ (when the circle touches AB),

next $AK \pm i$ times a continually increasing quantity,

and it will be readily perceived that the ' *shadow* ' circle that is to give us the roots cannot be the mere reflection of the other, but one that will accommodate itself to its *real* but retiring companion. Thus it will be found that for such an equation as $x^2 - 3x + 5^2 = 0$, the radius of the shadow circle is 5, and its centre $\frac{3}{2}$ below AB, and generally, for the equation

$$x^2 - ax + \left(\frac{a}{2} + k\right)^2 = 0,$$

the radius of the shadow circle is $\frac{a}{2} + k$, and the centre of the shadow circle is $\frac{a}{2}$ below AB, when AR and BS are made equal.

If, however, we adopt the more general and serviceable method of Fig. 6, taking $x^2 - 3x + 6 = 0$ as the equation, and making $AR = 2$, $BS = 3$, we at once see that the statement is not general enough. The modification needed is that if the radius of the real circle is r, the centre of the shadow circle must be taken at a distance of r on the other side of AB, so that for the general equation $x^2 - px + mn = 0$, the radius of the shadow circle is $\frac{1}{2}(m + n)$, and the centre of the shadow

circle is r below AB, r, the radius of the real circle, being equal to

$$\sqrt{\frac{p^2}{4} + \frac{(m-n)^2}{4}} \, .$$

This includes the special case considered above, and in every case the general reading of imaginary roots by the circle
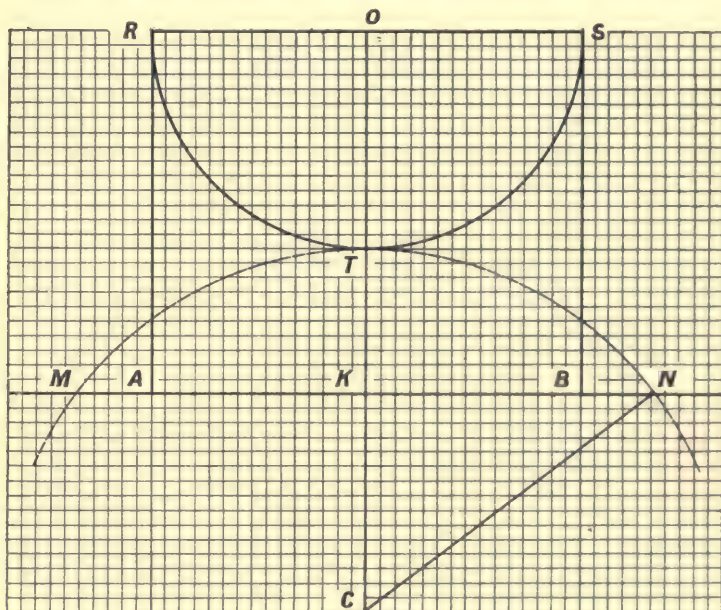

Fig. 5.

method is $AK \pm i \cdot KN$. The *shadow* circle is observed to meet the needs of its *real* comrade by assuming for its radius a length equal to the radius of its companion + the distance which the tangential point of the real circle moves away from AB, and by cutting AB in points M and N which recede

2—2

farther and farther from K as the real circle retires, thus giving the increasing multiple of i in the roots demanded by the increase in the absolute term mn. The general proof of the correctness of the method of reading off the roots for these two cases is appended.

For Fig. 5, the general equation may be represented by

$$x^2 - ax + b = 0,$$

and since the roots are

$$a/2 \pm \sqrt{a^2/4 - b},$$

$i \cdot KN$ should be

$$\sqrt{a^2/4 - b}.$$

Now in the figure we have

$$AB = a,$$
$$AR = BS = \sqrt{b},$$

and from its construction it follows that

$$CK = TO = a/2,$$
$$CN = CT = KO = \sqrt{b},$$
$$\therefore i \cdot KN = i \sqrt{CN^2 - CK^2}$$
$$= i \sqrt{b - a^2/4} = \sqrt{a^2/4 - b}$$

and

$$AK = a/2,$$
$$\therefore AK \pm i \cdot KN = a/2 \pm \sqrt{a^2/4 - b}.$$

For Fig. 6 the general equation may be represented by

$$x^2 - px + mn = 0,$$

and the roots being

$$p/2 \pm \sqrt{p^2/4 - mn},$$

$i \cdot KN$ should be

$$\sqrt{p^2/4 - mn}.$$

In the figure, $AB = p$, $AR = n$, $BS = m$, and, drawing a perpendicular from R to KO, it is seen that

radius of original circle $= \sqrt{(KO - AR)^2 + AK^2}$

$$= \sqrt{(m-n)^2/4 + p^2/4} = CK,$$
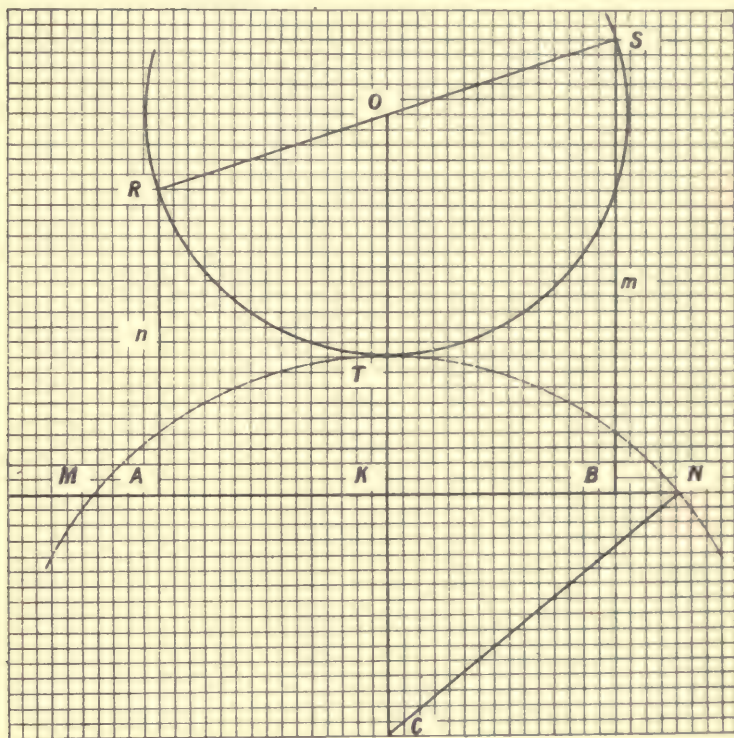
and $\qquad CN = CT = KO = \tfrac{1}{2}(m+n),$

Fig. 6.

$$\therefore i \cdot KN = i\sqrt{CN^2 - CK^2}$$

$$= i\sqrt{(m+n)^2/4 - [(m-n)^2/4 + p^2/4]}$$

$$= i \cdot \sqrt{mn - p^2/4} = \sqrt{p^2/4 - mn},$$

and $\qquad AK = p/2,$

$$\therefore AK \pm i \cdot KN = p/2 \pm \sqrt{p^2/4 - mn}.$$

III.

Intersection of a line and a circle in imaginary points.

The graphical determination of the imaginary points of intersection of a straight line and a circle is an immediate consequence of the foregoing. A typical case is represented in Fig. 7 for the circle

$$x^2 + y^2 = 1,$$

and the straight line $\qquad y = \tfrac{3}{4}x + 3.$

The general method is :—From centre O draw to given line AB a perpendicular OK cutting the circle at T, and set off KC equal to radius of given circle and on the other side of the line. C is the centre of the shadow circle, CT the radius, and M and N the imaginary points of intersection, their co-ordinates being read off in the form $a \pm ib$. The values are

For x. Abscissa of $K \pm i$ (difference of abscissae of K and N).

For y. Ordinate of $K \pm i$ (,,   ordinates   ,,   ).

Graph reading on inch paper gives

$$x = -1\cdot44 \pm 1\cdot74\,i,$$
$$y = +1\cdot94 \pm 1\cdot29\,i,$$

which correspond nearly to the calculated results,

$$x = -1\cdot44 \pm 1\cdot746i$$
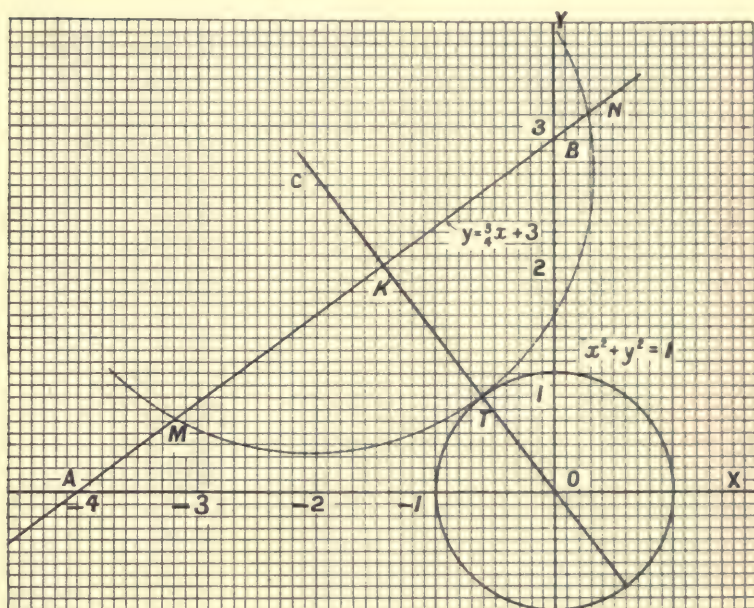$$y = +1\cdot92 \pm 1\cdot309i$$

FIG. 7.

The radius, co-ordinates of the centre, and the equation of the shadow circle are readily found by calculation, and its x-points of intersection with the straight line determined as

$$-1\cdot44 \pm \cdot16 \sqrt{119},$$

the x-points of intersection of the *real* circle and the straight line being $-1\cdot44 \pm \cdot16 \sqrt{119}i$.

Intersection of two circles in imaginary points.

Proceeding on somewhat similar lines with two equal circles, keeping one fixed with its centre at the origin and moving the other away with its centre along the x-axis, the locus of the imaginary points of intersection is found to be a rectangular hyperbola; on bringing the movable circle back into contact again with the fixed circle and then pushing it on to the left, the locus is at first, of course, the fixed circle for the real values and then the other branch of the hyperbola for the imaginaries. This is interesting but does not suggest the construction we are in search of. Now, since the common chord always passes through the points of intersection for the 'real' case, its representative ought to do duty for the imaginaries. This is clearly the radical axis, and the radical axis meets all the demands made on it. Making it serve as the AB line in the circle method for roots of quadratics, and getting shadow circles in a similar way for each of the real ones, it is found that the intersections of *either* shadow circle with the radical axis give the required points of intersection. This is illustrated by Fig. 8, the circles there represented being

$$(x-1)^2 + (y-1)^2 = 1^2 \} \ A$$
$$(x-4)^2 + (y-4)^2 = 2^2 \} \ B$$

MN is the radical axis and K its point of intersection with the centre line O_1O_2. C_1, the centre of A's shadow circle, is found by setting off KC_1 equal to radius of A. Similarly for
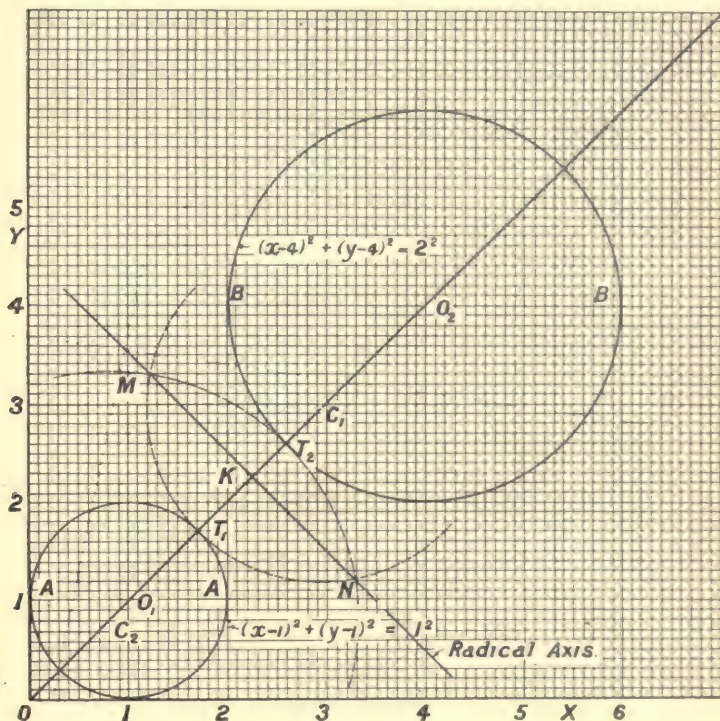

FIG. 8.

C_2. C_1T_1 and C_2T_2 are the radii of the shadow circles. Readings much as before.

For x. Abscissa of $K \pm i$ (difference of abscissae of N and K).

For y. Ordinate of $K \pm i$ (,,   ordinates   ,,   ).

By calculation $\quad x = 2{\cdot}25 \pm 1{\cdot}031i,$

$\qquad\qquad\qquad y = 2{\cdot}25 \mp 1{\cdot}031i.$

These statements are easily verified by finding the equations of the shadow circles and of the radical axis and obtaining their points of intersection by Algebra. In the case of *each* shadow circle, the equations giving the required points reduce to

$$\left.\begin{array}{c} x + y = \frac{9}{2} \\ x^2 - \frac{9}{2}x + 4 = 0 \end{array}\right\} .$$

Tangent to a circle from a point within and its imaginary points of contact.

A method of procedure is again suggested here by the principle of continuity. Starting with the usual method of drawing a tangent to a circle with centre O from an external point P we construct a circle on OP as diameter, and the intersections of this circle with the original give us the two points of contact. Imagine P to move up to the circle along a centre line. The tangential points gradually approach one another and coincide at the moment when P reaches the circle. Now if P continues its course towards the centre of the circle it is reasonable to suppose, from our previous work, that the tangential points should begin to recede from one another, probably at a greater rate, along a perpetually accommodating companion circle, and just as, in the common case of a tangent from an external point, the points of intersection of the real circles give us the tangential points, so we expect our two shadow circles to do us a like service when the given point is internal and the points of intersection imaginary. So they do and they appear always to intersect on the radical axis of the two real circles. But now a disturbing element comes in. Just as the join of the external P with a tangential point touches the real circle, we may reasonably expect that when P becomes an internal point its join with an imaginary tangential point will touch the shadow of the original circle.

Compasses and eyes give sufficient testimony on this score, and calculation confirms by proving that *both* the shadow circles are *cut* by the line we have been imagining to be a tangent to one of them. Further investigation establishes the results summarised below. It will be convenient however to precede this summary with a description of the figure (Fig. 9). A is the given circle with centre O, and P the internal point from which tangents to A are to be drawn ; B is the circle with OP as diameter [A is taken with a radius of 3, its equation being $x^2 + y^2 = 9$, and P is here taken $2\frac{1}{4}$ from O ; OP makes an angle of $45°$ with the x-axis], MN is the radical axis of A and B. The shadow circles of A and B are obtained in the usual way, KC_2 being equal to A's radius, and KC_1 equal to radius of B, M and N are the points of intersection of the shadow circles with the radical axis, and K the point where the centre line through P cuts the radical axis.

(1) The radical axis of the circles A and B is the polar of P with respect to the original circle A*, and, however purely imaginary the imaginary points of tangency are, Coordinate Geometry requires the polar to pass through them. The equation of the polar is $x + y = 4\sqrt{2}$, the tangential points are $2\sqrt{2} \pm \frac{1}{2}\sqrt{14}i$, $2\sqrt{2} \mp \sqrt{14}i$ and, dropping the i, these are the points given by the intersections, with the polar MN, of both the shadow circles.

(2) If the radius of the original circle A is a, and

       ,,   ,,   ,,   inner   ,, B ,, b,

* For, if radii of A and B (Fig. 9) are a and b respectively, O_1 the centre of B, and KM the polar of P with respect to A, then

$$\because OP \,.\, OK = a^2, \quad \therefore OK = a^2/2b, \quad \therefore OK^2 - O_1K^2 = (a^2/2b)^2 - (a^2/2b - b)^2$$
$$= b\,(a^2/b - b) = a^2 - b^2$$

            - = difference of squares on radii of A and B,

$\therefore KM$ is also the radical axis.

29

Fig. 9.

and the distance of the polar from the centre of A is d, then the radius of A's shadow circle is d, and the radius of B's shadow circle is $d - b$.

(3) The centre of the circle actually touched, which, for our present purpose, we will call the *tangent circle*, is always twice as far from centre of original circle as the polar is, so that $KC_3 = OK = d$,

and the radius of the tangent circle is $\sqrt{2d^2 - a^2}$.

It is easily proved that B's shadow circle passes through the centre of the tangent circle.

(4) The tangential points M and N can be obtained in several ways :—

I.   Along the polar of P set off $KM = KR$.

II. Along the centre line set off KC_2 equal to a, the radius of original circle, and with C_2 as centre and C_2V as radius obtain A's shadow circle.

III. •Along centre line set off KC_1 equal to b, the radius of B, and with C_1 as centre and C_1P as radius obtain B's shadow circle.

(5) KM is the inverse of the circle B with respect to A and also the inverse of B's shadow with respect to the tangent circle.

(6) The following results are given for convenience of reference :—

$\begin{cases} C_1 \text{ centre of } B\text{'s shadow,} \\ C_2 \quad\text{,,}\quad\text{,, } A\text{'s ,,} \\ C_3 \quad\text{,,}\quad\text{,, tangent circle.} \end{cases}$

$\begin{cases} KC_1 = b = \text{radius of circle } B, \\ KC_2 = a = \quad\text{,, ,, ,, } A, \\ KC_3 = d = \text{distance of polar from } A\text{'s centre.} \end{cases}$

$$\begin{cases} PC_1 = d - b = (2d^2 - a^2)/2d = \text{radius of } B\text{'s shadow,} \\ VC_2 = OK = KC_3 = d \qquad = \quad ,, \quad ,, \ A\text{'s} \quad ,, \\ MC_3 = \sqrt{2d^2 - a^2} \qquad\qquad = \quad ,, \quad ,, \ \text{tangent circle.} \end{cases}$$

$$KM = \sqrt{d^2 - a^2}.$$

$$PM = \sqrt{(d^2 - a^2)(2d^2 - a^2)/d^2} \ (\text{actual length}).$$

$$VK = d - a.$$

$$PK = d - 2b = (d^2 - a^2)/d.$$

$$PC_3 = (d^2 - a^2)/d + d = (2d^2 - a^2)/d.$$

The relation between the various circles is seen by considering how they change as P moves from V to O, the circle A being fixed.

In all positions KM is always equal to KR, and KC_3 to OK, and the rect. $OP . OK$ is constant, being always equal to a^2.

When P is at V, then R, K and M are also at V, and the two shadow and the tangent circles have a minimum value, C_2 and C_3 coinciding at a distance of a from K and their circles being each equal to the original circle A, while C_1 is at its maximum distance from K although its circle has a minimum area, its radius being half the radius of the original circle A.

As P moves towards the centre O, R also moves to the left along the circle A; K (and with it M) and the three centres C_1, C_2, C_3 move to the right at various rates, C_1 getting nearer and nearer to K, C_2 and C_3 further and further away. All this while C_2 moves exactly at the same rate as K, the distance between them always being a, the radius of the original circle, but the rate of C_3 is *double* that of K, the

distances of K from V being represented by 0, 1, 2, 3 ...... n, and the corresponding distances of C_3 from V by

$$1, 3, 5, 7 \ \ldots\ldots\ (2n-1),$$

the radius of the original circle being unity.

Suppose K to have moved on till it is 1000 units from O (radius of A, 1 unit),

A's shadow circle, centre C_2, will have a radius of 1000 exactly.

B's shadow circle, centre C_1, will have a radius of

$$(1000 - \tfrac{1}{2} \cdot \tfrac{1}{1000}).$$

Tangent circle, centre C_3, will have a radius of

$$1000\sqrt{2}\sqrt{1 - \tfrac{1}{2} \cdot \tfrac{1}{1000^2}}.$$

Relation between a real circle and its growing shadow.

(1) P and K (Fig. 10) are points in the centre line of circle A, radius a, such that $OP . OK = a^2$, $KC = OR = a$, and
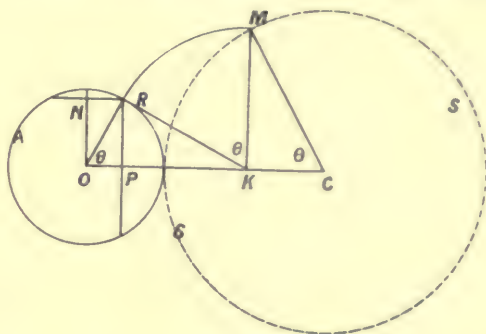
Fɪɢ. 10.

circle S with centre C touches A at the point V. S is there-fore the shadow circle of A.

From the construction, however obtained, the following conclusions are readily deduced :—

(a) P and K are inverse points with respect to the circle A.

(b) The triangle $KCM \equiv$ the triangle ROK.

H.   3

(c)   The triangle *KCM* is similar to the triangle *POR*.

(d)   The angles marked θ are equal.

Now if we suppose a point to start from V and to move either towards or away from O, the centre of the real circle A, the following relation always holds for the real circle and its shadow.   Denoting by d the distance at any moment of the point P (or of K if the movement is outwards) from the centre O, and speaking of PR (or KM) as the perpendicular, then *for the real case* (the one always considered in Elementary Geometry) the perpendicular is $\sqrt{a^2 - d^2}$, a real quantity, and *for the imaginary case* (that in which the geometrical construction is usually said to break down) the perpendicular is $\sqrt{d^2 - a^2}$, and since $i\sqrt{d^2 - a^2} = \sqrt{a^2 - d^2}$, it follows that if we let the same algebraical formula, $perp. = \sqrt{a^2 - d^2}$, which is true for the special case, hold generally, the shadow circle invariably gives the geometrical solution, $i \cdot KM$ corresponding strictly to the algebraical result.   This is illustrated later by one or two geometrical problems.

(2)   *For a real circle A, radius a,*

   the radius of the shadow circle $= a \sec \theta$,

for $CM = KC \sec \theta = a \sec \theta.$

Consider the changes in the triangles POR, KCM as P moves from V to centre O.

In every position $KC = a = OR$.

When P is at V, then K, R and M are also at V and the angles marked θ are $0°$.

As P moves towards O, K moves outwards,

   θ increases from $0°$ to $90°$, and $\sec \theta$ from 1 to ∞,

   $\therefore$ radius of shadow circle increases from a to ∞.

(3) *The parts of the polars of P and K* intercepted by the real circle and its shadow are as the radii of the circles,*

i.e. as $\qquad a : a \sec \theta,$ or as $1 : \sec \theta.$

$$\therefore KM = PR \sec \theta.$$

Limiting cases : P at V. $KM = PR \sec \theta = 0 (1) = 0.$

$\qquad\qquad\qquad P$ at O. $KM = a \sec \theta \qquad = \infty .$

(4) *The locus of M is a hyperbola,* P moving from V to v, and M denoting a point of intersection of A's shadow circle with the polar of P (Fig. 11); vV is the diameter of A.

FIG. 11.

Taking as axes OX, OY at angles of $45°$ with OK, it is easy to show that the rect. $OS \cdot SM$ is constant for all positions of M. For, if the circle MRN is described with K as centre, the triangle NKM is obviously isosceles.

Also the triangle OKY is isosceles, angle at O being $45°$.

* Here and throughout, always with reference to the original circle A, unless otherwise stated.

On producing MN to S, it is seen that the angle S is $90°$, and the triangle OSN equal to the right-angled triangle MTY.

Also the rect. $OS \cdot SM = ON\,MY$

$$= \text{triangle } OKY - \text{triangle } NKM$$
$$= \tfrac{1}{2}OK^2 - \tfrac{1}{2}NK^2$$
$$= \tfrac{1}{2}(OK^2 - RK^2)$$
$$= \tfrac{1}{2}OR^2$$
$$= \tfrac{1}{2}a^2.$$

The equation of the curve is therefore $xy = \tfrac{1}{2}a^2$, a hyperbola referred to the asymptotes OX, OY as axes, and with V as vertex. When P passes the centre O, KM appears on the left and we get the other branch of the hyperbola with its vertex at v. Referred to OC, OD as axes, the equation of the hyperbola is $x^2 - y^2 = a^2$, the equation of the circle A being $x^2 + y^2 = a^2$, also referred to OC, OD as axes.

If, in the light of this, we now reconsider the question raised on p. 24 as to the points of intersection of two equal circles, one with its centre fixed at the origin, equation $x^2 + y^2 = a^2$, the other moving right and left with its centre always on the x-axis, we shall see that the locus of the points of intersection is always given by the equation

$$x^2 \pm (iy)^2 = a^2.$$

Application to Elementary Geometry.

1. *Divide* 10 *into two parts so that their product shall be* 40.

This problem is given by Cardano (1545) and is interesting as being apparently the earliest on record to start the quest after imaginaries (see Chrystal's *Algebra*, Part I. p. 248, Historical Note).

A geometrical solution, answering to the algebraical, can be obtained by help of shadow circle thus :—

Construct Fig. 6, making AB, AR, BS, 10, 5, 8 half-inches respectively, and read off from shadow circle as on p. 19. The calculated result is $5 \pm \sqrt{15}i$, or $5 \pm 3\cdot873i$ approx.

2. A similar example and fairly typical of the algebraic geometrical problems suggested by Euc. II. 11 and 14, is

Divide a line 4 *inches long into two parts so that the sum of the squares on them shall be* 4 *sq. ins.*

Denoting the parts by x and $4 - x$ we get

$$x^2 - (4 - x)^2 = 4,$$

i.e. $x^2 - 4x + 6 = 0,$

which is worked as in Fig. 6, the roots being read as

$$2 \pm 1\cdot41i.$$

3. *Given a circle, radius a, find length of chord which is distant d from the centre,* (i) *when $d < a$,* (ii) *when $d > a$.*

Denoting the length of half the chord by z, then for the ordinary case in which the distance d from the centre is less than the radius a, we have $z = \sqrt{a^2 - d^2}$, and the corresponding geometrical construction is to set off $OP = d$ along a centre line OC (Fig. 10) and to draw the perpendicular PR to meet the circle, in which case $PR = z = \sqrt{a^2 - d^2}$.

If we now inquire what is the geometrical significance of the question when the lengths have been so chosen that $d > a$, the reply is that although we can find no chord *within* the given circle to satisfy the conditions, any more than we can, by keeping to the primitive meaning of the terms, 'divide a straight line 5 cm. long into two parts whose difference is 7 cm.,' yet just as in the latter case a geometrical construction can be found in harmony with the algebraical statement of the case, so in the case under present consideration can geometry, by means of the shadow circle, be made to harmonise with its fleeter companion Algebra.

The construction is just as in the 'real' case, OK, Fig. 10, being set off equal to d, and the length of the perpendicular KM that meets the shadow circle, when multiplied by i answers fully to the algebraical result for

$$KM = \sqrt{CM^2 - KC^2}$$
$$= \sqrt{d^2 - a^2},$$
$$\therefore i \cdot KM = \sqrt{a^2 - d^2} = z, \text{ just as before.}$$

4. *Given a circle, radius a, find how far distant from the centre is a chord 2c units in length,* (i) *when c < a,* (ii) *when c > a.*

Worked much as last example; Fig. 10 illustrates the method. For the ordinary case, twice RN is the chord of length $2c$, ON or PR is the required distance.

For the 'imaginary' case, set off as before a length equal to c; the point P (now shown as K in the figure) falls outside the given circle, therefore have recourse to the shadow circle and $i \cdot KM$ gives the required length.

5. *Euc.* II. 5, 6 *and* 14 *and the introduction of the imaginary.*

When Euc. II. 5 and 6 are looked at with modern eyes accustomed to negatives and imaginaries, and account taken of the 'sense' of the lines, then instead of being regarded as two distinct propositions their identity is seen to be closer even than geometry books have often indicated when giving their algebraical representation. This essential likeness would suggest that when we have a geometrical construction, as in II. 14, based on the more primitive conception of II. 5 and giving us a positive root of $\sqrt{(+a)(+b)}$, then the introduction of the negative idea, besides making the old construction yield us the negative root, would give completion by leading us on in due course to the imaginary. The corresponding representation of the imaginary roots of $\sqrt{(+a)(-b)}$ is the inevitable consequence, and readily follows with the aid of II. 6, the negative twin brother of II. 5, and of the shadow circle with its power of adapting itself to all conceivable changes in a and b. Not only will the real identity of these two propositions be observed in what follows, but the construction for imaginary roots will be seen to follow closely in the steps of that for real roots, justified too with an almost identical proof.

Fig. 12 suggests the usual construction and proof of Euc. II. 14 by means of II. 5, but for comparison with the companion problem for imaginary roots the following observations may be made :—

$$DP = a, \quad PE = b \text{ (Fig. 12)}.$$

The diameter DE of the circle with centre $O = a + b$,

$$\therefore \text{ the radius } OR = \tfrac{1}{2}(a + b),$$
$$\text{and } OP = \tfrac{1}{2}(a - b).$$
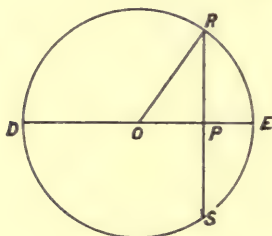
Fig. 12.

The perpendicular from which the roots are read is drawn from P, the common extremity of a and b, PR giving the positive and PS the negative root.

The actual length of $PR = \sqrt{OR^2 - OP^2}$

$$= \sqrt{\tfrac{1}{4}(a + b)^2 - \tfrac{1}{4}(a - b)^2}$$
$$= \sqrt{ab} = \sqrt{(+a)(+b)}.$$

Fig. 13 shows the construction for the 'imaginary' case. Using the same letters as before, $\quad DP = a, \; PE = -b.$

The diam. DE of the circle with centre $O = a + (-b)$;

$$\therefore \text{ the radius } OR_1 = \tfrac{1}{2}[a + (-b)] = \tfrac{1}{2}(a - b),$$
$$\text{and } OP = \tfrac{1}{2}[a - (-b)] = \tfrac{1}{2}(a + b).$$

The perp. from which the roots are read is drawn from P, the common extremity of a and b, PR as before giving the

positive and PK the negative root. Here, however, P is outside the real circle, centre O, so the shadow circle is called
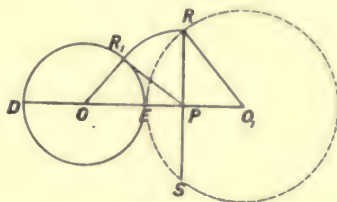
Fig. 13.

into requisition, its centre O_1 being determined by setting off, along the centre line, PO_1 equal to radius of real circle, its radius being O_1E.

It is at once evident that

$$O_1R = EO_1 = OP = \tfrac{1}{2}(a+b)$$

and

$$PO_1 = OR_1 = \tfrac{1}{2}(a-b).$$

Hence the actual length of $PR = \sqrt{O_1R^2 - PO_1^2}$

$$= \sqrt{\tfrac{1}{4}(a+b)^2 - \tfrac{1}{4}(a-b)^2},$$
$$\therefore i \cdot PR = \sqrt{\tfrac{1}{4}(a-b)^2 - \tfrac{1}{4}(a+b)^2}$$
$$= \sqrt{(+a)(-b)}.$$

In short, for the 'imaginary' case the only difference is that in the algebraical part $-b$ is substituted for $+b$, while in the geometrical construction the perpendicular is drawn to meet the shadow instead of the real circle with the consequent reading of $\pm i \cdot PR$ for the result*.

* To complete this it may be added that an apt illustration of the geometrical significance of $\sqrt{(-a)(-b)}$ is readily given by a piece of tracing paper. If Fig. 12 (segments DP, PE being $+a$, $+b$ is marked through on tracing paper and then swung about the point P through an angle of $180°$ the effect of the swing is to negative both a and b, and the lines representing the roots are seen to coincide with those of $\sqrt{(+a)(+b)}$ so that $\sqrt{(-a)(-b)} = \sqrt{(+a)(+b)}$.

The Elements of Geometry

By R. LACHLAN, Sc.D.,

Formerly Fellow of Trinity College, Cambridge,

AND

W. C. FLETCHER, M.A.,

Chief Inspector of Secondary Schools.

With about 750 Exercises and Answers. Clearly printed and furnished with a large number of bold and distinct Diagrams.

Crown 8vo. pp. xii + 207. 2s. 6d.

This book has been prepared in accordance with the recommendations of the Mathematical Committee of the British Association, and on the lines laid down in the Syllabus lately issued by Cambridge University for the Local Examinations.

Elementary Geometry

Containing the whole substance of Euclid I.-IV., IV., except the elegant but unimportant Proposition IV. 10.

By W. C. FLETCHER, M.A.,

Chief Inspector of Secondary Schools, late Fellow of St John's College, Cambridge.

84 pages. Crown 8vo. Cloth. Price 1s. 6d.

The Mathematical Gazette.--"We cannot speak too highly of Mr Fletcher's work. It has been constructed with admirable skill, and every teacher will find it most suggestive."

Nature.—"This is a very small book and a very good one."

Plane Geometry

Adapted to Heuristic Methods of Teaching.

By T. PETCH, B.A., B.Sc.,

Lecturer in Mathematics, Leyton Technical Institute.

viii + 112 pages. Cloth, 1s. 6d.

LONDON : EDWARD ARNOLD.

Five-Figure Tables of Mathematical Functions

COMPRISING

Tables of Logarithms, Powers of Numbers, Trigonometric, Elliptic, and other Transcendental Functions.

By JOHN BORTHWICK DALE, M.A.,

Late Scholar of St. John's College, Cambridge,
Lecturer in Pure and Applied Mathematics in King's College, University of London

Demy 8vo. Cloth. 3s. 6d.

This collection of Tables has been formed chiefly with a view to
meet the requirements of workers in Physical Science and Applied
Mathematics. It includes many tables which have not hitherto
found a place in such collections, while those tables which are of
use only in Navigation have been excluded.

ELEMENTARY PLANE AND SOLID MENSURATION

For use in Schools, Colleges, and Technical Classes.

By R. W. K. EDWARDS, M.A.,

Formerly Scholar of Jesus College, Cambridge,
Lecturer in Mathematics at King's College, University of London.

xii + 322 pages. Crown 8vo., cloth, 3s. 6d.

CONTENTS.

Preface.— Useful Tables.— I. Use of Mathematical Tables.—
II. Parallelograms.—III. Triangles.—IV. Trapezoids.—V. Regular Polygons.—VI. Similar Figures.—VII. Irregular Rectilineal
Figures.—VIII. The Circle.—IX. Easy Graphs.—X. Simpson's
Rule.—XI. Parallelopipeds.—XII. Prisms.—XIII. Pyramids.—
XIV. Cylinders.—XV. Cones.—XVI. Similar Solids.—
XVII. Frusta.—XVIII. The Sphere.—Appendix.—Index.—
Answers.

Educational Times. —"An excellent little volume. With the knowledge
and experience of a teacher, the author writes so as to reproduce in
the reader's mind a vivid image of the thoughts clearly arranged in
his own."

School World.—"A useful and well-written treatise, in which the
rules are proved as well as enunciated."

LONDON : EDWARD ARNOLD.

Graphical Statics Problems.

With Diagrams. By W. M. BAKER, M.A., Headmaster of the Military and Civil Department at Cheltenham College. 4to., 2s. 6d.

The Principles of Mechanism.

Being a Short Treatise on the Kinematics and Dynamics of Machines. By HERBERT A. GARRATT, Assoc. M. Inst. C.E., Head of the Engineering Department of the Northern Polytechnic Institute, Holloway. viii + 166 pages. Crown 8vo., cloth, 3s. 6d.

Vectors and Rotors.

With Applications. Being Lectures delivered at the Central Technical College. By Professor O. HENRICI, Ph.D., F.R.S., LL.D., and G. C. TURNER, B.Sc., Goldsmiths' Institute. Crown 8vo., cloth, 4s. 6d.

Nature.—"Professor Henrici can always be depended upon to embellish any mathematical subject which he touches, because with the skill of the analyst he combines the keen perception of the geometer, which ever seeks to render the results of analysis in some way visible by spatial representation. To a mathematician of this kind the subject of vector analysis is peculiarly appropriate. We are therefore indebted to Mr Turner for putting into systematic form the lectures delivered by Professor Henrici, and producing a very simple and elementary work the methods and ideas of which should find a very early introduction into our ordinary mathematical teaching."

Calculus for Engineers.

By JOHN PERRY, M.E., D.Sc., F.R.S.

Professor of Mechanics and Mathematics in the Royal College of Science, London; Vice-President of the Physical Society; Vice-President of the Institution of Electrical Engineers.

Fifth Edition. viii + 382 pages, with 106 Illustrations.

Crown 8vo., cloth, 7s. 6d.

The appearance of the fifth edition of this valuable text-book is the best proof of its usefulness to engineering students.

An Elementary Treatise on Practical Mathematics.

By JOHN GRAHAM, B.A., B.E.,

Demonstrator and Lecturer on Applied Mathematics in the City and Guilds of London Institute's Technical College, Finsbury. Author of "The Calculus for Engineering Students."

viii + 276 pages, with Answers and numerous Diagrams.

Crown 8vo., cloth, 3s. 6d.

F. BERRY, Esq., Technical School, Bolton.—"I think it a very useful book, and shall recommend it to my students."

W. W. PULLEN, Esq., South-Western Polytechnic Institute, London.—"Such a work has been wanted for some time for engineers, both mechanical and electrical. I shall have much pleasure in recommending it to my students."

THE LONDON SCHOOL ATLAS.

Edited by the Right Hon. H. O. ARNOLD-FORSTER, M.P., Author of "The Citizen Reader," "This World of Ours," etc. A magnificent Atlas, including 48 pages of Coloured Maps, several of them double page, and Pictorial Diagrams. With an Introduction on the Construction and Reading of Maps by A. J. HERBERTSON, Ph.D.

Among the notable features of this Atlas are : (1) The Specimens of Ordnance Surveys and Admiralty Charts ; (2) the lucid Astronomical Diagrams ; (3) the beautifully-coloured Physical Maps ; (4) the careful selection of names without overcrowding ; (5) the constant presentation of uniform scales for comparison ; (6) a Historical Series of Maps illustrating the Building of the British Empire ; (7) an excellent Map of Palestine.

The size of the Atlas is about 12 by 9 inches, and it is issued in the following editions :

Stout paper wrapper, with cloth strip at back, 1s. 6d.	Cloth cut flush, 2s. 6d.
	Limp cloth, 3s.
Paper boards, 2s.	Cloth gilt, bevelled edges, 3s. 6d.

A Manual of Physiography.

By ANDREW HERBERTSON, Ph.D., F.R.G.S., Reader in Regional Geography at the University of Oxford. Fully Illustrated. Cloth, 4s. 6d.

Arnold's New Shilling Geography.

The World, with special reference to the British Empire. 160 pp. Crown 8vo., cloth, 1s.

The New Zealand Colony.

Its Geography and History. With Illustrations and a Coloured Map. Cloth, 1s.

The Australian Commonwealth.

Its Geography and History. A reading book for Schools. 144 pages. With Illustrations and Coloured Map. Cloth, 1s.

A Historical Geography.

By the late Dr. MORRISON. New Edition, revised and largely re-written by W. L. CARRIE, Headmaster at George Watson's College, Edinburgh. Crown 8vo., cloth, 3s. 6d.

The Shilling Geography.

By the late Dr. MORRISON. New Edition, revised by W. L. CARRIE. Small crown 8vo., cloth, 1s.

Arnold's Geographical Handbooks.

A Series of little Manuals providing accurate and clearly arranged summaries of Geographical information. 3d. each ; cloth, 6d.

1. England and Wales.
2. Scotland, Ireland, Canada, and Australia.
3. Europe.
4. Handbook to Greater Britain.
5. British Isles.

6. Asia.
7. Africa.
8. United States
9. Central and South America
10. The World in Outline

ARNOLD'S SCHOOL SHAKESPEARE.

General Editor : J. CHURTON COLLINS, M.A.

Price 1s. 3d. each.

Macbeth. By R. F. CHOLMELEY, M.A., Assistant Master at S. Paul's School.

Twelfth Night. By R. F. CHOLMELEY, M.A.

As You Like It. By S. E WINBOLT, B.A., Assistant Master at Christ's Hospital.

Julius Cæsar. By E. M. BUTLER, B.A., Assistant Master at Harrow School.

Midsummer Night's Dream. By R. BRIMLEY JOHNSON, Editor of Jane Austen's Novels, etc.

The Merchant of Venice. By C. H. GIBSON, M.A., late Assistant Master at Merchant Taylors' School.

The Tempest. By W. E. URWICK, Lecturer on Modern Languages in Durham University.

Price 1s. 6d. each.

King Lear. By the Rev. D. C. TOVEY, M.A., late Assistant Master at Eton College.

Richard II. By C. H. GIBSON, M.A.

Henry V. By S. E. WINBOLT, B.A.

Richard III. By F. P. BARNARD, M.A., late Headmaster of Reading School

King John. By F. P. BARNARD, M.A.

Coriolanus. By R. F. CHOLMELEY, M.A.

Hamlet. By W. HALL GRIFFIN, Professor of English Literature at Queen's College, London.

ARNOLD'S BRITISH CLASSICS FOR SCHOOLS.

Issued under the General Editorship of J. CHURTON COLLINS, M.A.

Paradise Lost, Books I. and II. By J. SARGEAUNT, M.A., Assistant Master at Westminster School. Cloth, 1s. 3d.

Paradise Lost, Books III. and IV. By J. SARGEAUNT, M.A. 1s. 3d.

Marmion. By G. TOWNSEND WARNER, M.A., Fellow of Jesus College, Cambridge, and Assistant Master at Harrow School. Cloth, 1s. 6d.

Macaulay's Lays of Ancient Rome. Assistant Master at Winchester College.

The Lay of the Last Minstrel. By G. TOWNSEND WARNER, M.A. 1s. 3d.

The Lady of the Lake. By J. MARSHALL, M.A., Rector of the Royal High School, Edinburgh. Cloth, 1s. 6d.

Childe Harold. By the Rev. E. C. EVERARD OWEN, M.A., Assistant Master at Harrow School. Cloth, 2s.

By R. L. A. DU PONTET, M.A., Cloth, 1s. 6d.

SELECTIONS FROM THE POEMS OF TENNYSON. Edited, with Introduction and Notes, by the Rev. E. C. EVERARD OWEN, M.A., Assistant Master in Harrow School. Crown 8vo., cloth, 1s. 6d.

PATRIOTIC SONG. A Book of English Verse. Being an Anthology of the Patriotic Poetry of the British Empire from the Defeat of the Spanish Armada until the Death of Queen Victoria. xxvii + 363 pages. Crown 8vo., 2s. 6d.

A FIRST COURSE IN ENGLISH LITERATURE. By RICHARD WILSON, B.A. 144 pages. Crown 8vo., price 1s.

A FIRST COURSE IN ENGLISH ANALYSIS AND GRAMMAR. By RICHARD WILSON, B.A., Author of "Arnold's Language Lessons." 144 pages. Crown 8vo., cloth, 1s.

LONDON : EDWARD ARNOLD, 41 & 43 MADDOX STREET, W.

LITERARY READING BOOKS.

IN GOLDEN REALMS. An English Reading Book for Junior
Forms. 224 pages. Crown 8vo., cloth, 1s. 6d.

Designed to form an introduction to the study of English literature. Containing
folk-tales from various sources, stories from Homer, Virgil, the Book of poems,
Chaucer, Malory, Froissart, Spenser, Shakespeare, etc., etc. Illustrated with
beautiful black-and-white reproductions of twelve famous paintings.

IN THE WORLD OF BOOKS. An English Reading Book for
Middle Forms. 256 pages. Crown 8vo., cloth, 1s. 6d.

Containing interesting extracts, generally narrative, from the leading writers of
prose and poetry from Caedmon and Bede to Thackeray and Tennyson, arranged in
chronological order. Illustrated in a manner similar to the above.

THE GREENWOOD TREE. A Book of Nature Myths and Verses.
224 pages. Illustrated, 1s. 3d.

LAUREATA. Edited by RICHARD WILSON, B.A. Crown 8vo., cloth,
1s. 6d. Beautifully printed and tastefully bound.

A collection of gems from the best poets from Shakespeare to Swinburne.

TELLERS OF TALES. Edited by RICHARD WILSON, B.A. Crown
8vo., cloth, 1s. 6d.

Biographies of some English novelists, with Extracts from their works.

POETS' CORNER. Selected Verses for young Children. Fcap. 8vo.,
cloth, 1s.

LATIN.

THE FABLES OF ORBILIUS. By A. D. GODLEY, M.A., Fellow
of Magdalen College, Oxford. With humorous Illustrations. Crown 8vo.,
cloth. Book I., 9d. ; Book II., 1s.

SENTENCES FOR LATIN COMPOSITION. Based upon the
Exercises in "Fables of Orbilius, Part II." By Rev. A. James Smith, M.A.,
Headmaster of King Edward's Grammar School, Birmingham. Cloth limp, 8d.

VIRGIL—ÆNEID. Books I., II., and III. The New Oxford Text,
by special permission of the University. Edited, with Introduction and Notes.
by M. T. TATHAM, M.A. Crown 8vo., cloth, 1s. 6d. each.

A FIRST LATIN COURSE. By G. B. GARDINER, M.A., D.Sc.,
and A. GARDINER, M.A. viii + 227 pages. 2s.

A Key, on Teachers' direct order only, 2s. net.

A SECOND LATIN READER. With Notes and Vocabulary. By
GEORGE B. GARDINER, M.A., D.Sc., and ANDREW GARDINER, M.A. Cloth, 1s. 6d.

A LATIN ANTHOLOGY FOR BEGINNERS. By George B.
GARDINER, M.A., D.Sc., and A. GARDINER, M.A. Crown 8vo., 2s.

FORUM LATINUM. A First Latin Book. By E. VERNON ARNOLD,
Litt.D., Professor of Latin at the University College of North Wales. In
three parts. 1s. 4d. each. Complete, 3s. 6d.

CÆSAR'S GALLIC WAR. Books I. and II. Edited by T. W.
HADDON, M.A., and G. C. HARRISON, M.A. With Notes, Maps, Plans, Illustra-
tions, Helps for Composition, and Vocabulary. Cloth, 1s. 6d.

Books III.-V. Edited for the use of Schools by M. T. TATHAM, M.A. Uniform with
Books I. and II. Crown 8vo., cloth, 1s. 6d.

Books VI. and VII. By M. T. TATHAM, M.A. Uniform with Books III.-V. 1s. 6d.

A LATIN TRANSLATION PRIMER. With Grammatical Hints,
Exercises and Vocabulary. By GEORGE B. GARDINER, Assistant Master at the
Edinburgh Academy, and ANDREW GARDINER, M.A. Crown 8vo., cloth, 1s.

Arnold's School Series.

GERMAN.

EASY GERMAN TEXTS. For pupils who have acquired a simple vocabulary and the elements of German. Under the General Editorship of WALTER RIPPMANN, M.A., Professor of German at Queen's College, London. With exercises on the text. Small crown 8vo., cloth, 1s. 3d. each.

ANDERSEN'S BILDERBUCH OHNE BILDER (What the Moon Saw).

PRINZESSIN ILSE. By MARIE PETERSEN.

DER TOPFER VON KANDERN. By H. VILLINGER.

DIE FLUT DES LEBENS. By ADOLF STERN.

GERMAN WITHOUT TEARS. By Mrs. HUGH BELL. A version in German of "French Without Tears." With illustrations. Cloth.
Part I., 9d.   Part II., 1s.   Part III., 1s. 3d.

LESSONS IN GERMAN. A graduated German Course, with Exercises and Vocabulary, by L. INNES LUMSDEN, late Warden of University Hall, St. Andrews. Crown 8vo., 3s.

EXERCISES IN GERMAN COMPOSITION. By RICHARD KAISER, Teacher of Modern Languages in the High School of Glasgow. Including carefully graded Exercises, Idiomatic Phrases, and Vocabulary. Cloth, 1s. 6d.

KLEINES HAUSTHEATER. Fifteen little Plays in German for Children. By Mrs. HUGH BELL. Crown 8vo., cloth, 2s.

GERMAN DRAMATIC SCENES. By C. ABEL MUSGRAVE. With Notes and Vocabulary. Crown 8vo., cloth, 2s. 6d.

FRENCH.

ELEMENTS OF FRENCH COMPOSITION. By J. HOME CAMERON, M.A., Lecturer in French in University College, Toronto, Canada. viii + 196 pages. Crown 8vo., cloth, 2s. 6d.

LE FRANCAIS CHEZ LUI. A French Reader on Reform Lines, with Exercises on Grammar for Middle and Junior Forms. By W. H. HODGES, M.A., Modern Language Master at Merchant Taylors' School, and P. POWELL, M.A., Assistant Master at Loretto School. Cloth, 1s. 3d.

MORCEAUX CHOISIS. French Prose Extracts. Selected and Edited by R. L. A. DU PONTET, M.A., Assistant Master in Winchester College. Explanatory Notes and Short Accounts of the Authors cited are given. Crown 8vo., cloth, 1s. 6d.

POEMES CHOISIS. Selected and Edited by R. L. A. DU PONTET, M.A. Cloth, 1s. 6d.

LES FRANCAIS EN MÉNAGE. By JETTA S. WOLFF. With Illustrations. Crown 8vo., cloth, 1s. 6d. An entirely original book, teaching the ordinary conversation of family life in France by a series of bright and entertaining scenes.

LES FRANCAIS EN VOYAGE. By JETTA S. WOLFF. A companion volume to the preceding, giving a lively account of travelling on the continent. Cleverly illustrated. Crown 8vo., cloth, 1s. 6d.

FRANCAIS POUR LES TOUT PETITS. By JETTA S. WOLFF. With Illustrations by W. FOSTER. Cloth, 1s. 3d.

LES FRANCAIS D'AUTREFOIS. Stories and Sketches from the History of France. By JETTA S. WOLFF. Cloth, 1s. 3d.

ARNOLD'S SCHOOL SERIES.

FRENCH.

FRENCH WITHOUT TEARS. A graduated Series of French Reading Books, carefully arranged to suit the requirements of quite young children beginning French. With Humorous Illustrations, Notes, and Vocabulary. By Mrs. Hugh Bell, author of "Le Petit Theatre Français." Crown 8vo., cloth, Book I., 9d. Book II., 1s. Book III., 1s. 3d.

A FIRST FRENCH COURSE. Complete, with Grammar, Exercises and Vocabulary. By James Boïelle, B.A. (Univ. Gall.), Senior French Master at Dulwich College, etc. Crown 8vo., cloth, 1s. 6d.

A FIRST FRENCH READER. With Exercises for Re-translation. Edited by W. J. Greenstreet, M.A., Head Master of the Marling School, Stroud. Crown 8vo., cloth, 1s.

FRENCH DRAMATIC SCENES. By C. Abel Musgrave. With Notes and Vocabulary. Crown 8vo., cloth, 2s.

SIMPLE FRENCH STORIES. An entirely new series of easy texts, with Notes, Vocabulary, and Table of Irregular Verbs, prepared under the General Editorship of Mr. L. Von Glehn, Assistant Master at Forse School, Cambridge. About 80 pages in each volume. Limp cloth, 9d.

Un Drame dans les Airs. By Jules Verne. Edited by I. G. Lloyd Jones, B.A., Assistant Master at Cheltenham College.

Un Anniversaire à Londres, and two other stories. By P. J. Stahl. Edited by C. E. B. Howitt, M.A., Assistant Master at Marlborough College.

Pif-Paf. By Edouard Laboulaye. Edited by W. M. Poole, M.A., Lecturer in French to the Channel Squadron.

Monsieur le Vent et Madame la Pluie. By Paul de Musset. Edited by Miss Leahy, Assistant Mistress at the Girls' High School, Sheffield.

La Petite Souris Grise; and Histoire de Rosette. By Madame de Ségur. Edited by Blanche Daly Cocking.

Poucinet, and two other tales. By Edouard Laboulaye. Edited by W. M. Poole, M.A.

GIL BLAS IN THE DEN OF THIEVES. Arranged from Le Sage. With Notes and Vocabulary by R. de Blanchaud, B.A., Assistant Master at the Central Schools, Aberdeen. Limp cloth, crown 8vo., 9d.

[Uniform with the above series.

L'APPRENTI. By Emile Souvestre. Edited by C. F. Hardener, French Master at Berkhamsted School. Crown 8vo., cloth, 1s.

RICHARD WHITTINGTON. By Madame Eugenie Foa. And UN CONTE DE L'ABBE DE SAINT-PIERRE. By Emile Souvestre. Edited by C. F. Hardener. Crown 8vo., cloth, 1s.

The feature of these two volumes is that in addition to the notes and vocabulary there is a set of exercises, chiefly in the form of questions and answers in French, modelled upon the text of the narrative. This innovation promises to prove very popular.

The following volumes are all carefully prepared and annotated by acid, well known editors as Messrs. F. Tarver, J. Boïelle, etc., and will be found thoroughly adapted for school use.

JULES VERNE VOYAGE AU CENTRE DE LA TERRE. 1s.
ALEXANDRE DUMAS LE MASQUE DE FER. 1s.
ALEXANDRE DUMAS VINGT ANS APRES. 3s.
FRENCH REVOLUTION READINGS. 3s.
P. J. STAHL MAROUSSIA. 2s.
EMILE RICHEBOURG LE MILLION DU PERE RACLOT. 1s.
H. de BALZAC UNE TÉNÉBREUSE AFFAIRE. 2s.
VICTOR HUGO QUATRE VINGT TREIZE. 3s.
ALEXANDRE DUMAS MONTE CRISTO. 3s.
HENRI GREVILLE PERDUE. 3s.

MATHEMATICS.

A New Arithmetic for Schools. By J. P. KIRKMAN, M.A., and A. E. FIELD, M.A., Assistant-masters at Bedford Grammar School. Crown 8vo., cloth, 3s. 6d.

Exercises in Arithmetic (Oral and Written). Parts I., II., and III. By C. M. TAYLOR (Mathematical Tripos, Cambridge), Wimbledon High School. Cloth, 1s. 6d. each.

Algebra. Part I., "The Elements of Algebra," including Quadratic Equations and Fractions. By R. LACHLAN, Sc.D., with or without Answers, 2s. 6d. Answers separately, 1s.

Algebra for Beginners. By J. K. WILKINS, B.A., and W. HOLLINGSWORTH, B.A. In Three Parts, carrying the pupil as far as Quadratic Equations. Part I., 4d.; Part II., 4d.; Part III., 6d. Answers to Parts I.-III., in one vol., 6d.

Vectors and Rotors. With Applications. Being Lectures delivered at the Central Technical College. By Professor O. HENRICI, F.R.S. Edited by G. C. TURNER, Goldsmith Institute. Crown 8vo., 4s. 6d.

An Elementary Treatise on Practical Mathematics. By JOHN GRAHAM, B.A., Demonstrator of Mechanical Engineering and Applied Mathematics in the Technical College, Finsbury. Cr. 8vo., cloth, 3s. 6d.

Preliminary Practical Mathematics. By S. G. STARLING, A.R.C.Sc., B.Sc., Head of the Mathematics and Physics Department of the West Ham Municipal Technical Institute; and F. C. CLARKE, A.R.C.Sc., B.Sc. 1s. 6d.

The Principles of Mechanism. By H. A. GARRATT, A.M.I.C.E., Head of the Engineering Department of the Northern Polytechnic Institute, Holloway. Crown 8vo., cloth, 3s. 6d.

Five-Figure Tables of Mathematical Functions. By J. B. DALE, M.A. Camb., B.A. Lond., late Scholar St. John's College, Cambridge, Lecturer on Pure and Applied Mathematics, King's College, University of London. Demy 8vo., 3s. 6d. net.

The Elements of Geometry. By R. LACHLAN, Sc.D., formerly Fellow of Trinity College, Cambridge, and W. C. FLETCHER, M.A. With about 750 Exercises and Answers. Cloth, 2s. 6d.

Elementary Geometry. By W. C. FLETCHER, M.A., Crown 8vo., cloth, 1s. 6d.

A First Geometry Book. By J. G. HAMILTON, B.A., Lecturer on Geometry at the Froebel Educational Institute; and F. KETTLE, B.A., Headmaster of Clapham High School for Boys. Crown 8vo., fully illustrated, cloth, 1s. Answers (for Teachers only), 6d.

Graphs and Imaginaries. By J. G. HAMILTON, B.A., and F. KETTLE, B.A. Cloth, 1s. 6d.

The Elements of Euclid. By R. LACHLAN, Sc.D.

THE FOLLOWING EDITIONS ARE NOW READY.

Book I. 145 pages, 1s.	Books I.—IV. 346 pages, 3s.
Books I. and II. 180 pages, 1s. 6d.	Books III. and IV. 164 pages, 2s.
Books I.—III. 304 pages, 2s. 6d.	Books I.—VI. and XI. 500 pages. 4s. 6d.
Books IV.—VI. 2s. 6d.	Book XI. 1s.

Mensuration. By R. W. K. EDWARDS, M.A., Lecturer on Mathematics at King's College, London. Cloth, 3s. 6d.

ARNOLD'S SCHOOL SERIES.

SCIENCE.

The Elements of Inorganic Chemistry. For use in Schools and Colleges. By W. A. SHENSTONE, F.R.S., Lecturer in Chemistry at Clifton College. xii + 506 pages. Cloth, 4s. 6d.

A Course of Practical Chemistry. By W. A. SHENSTONE. Cloth, 1s. 6d.

A First Year's Course of Experimental Work in Chemistry. By E. H. COOK, D.Sc., F.I.C., Principal of the Clifton Laboratory, Bristol. Crown 8vo., cloth, 1s. 6d.

A Text-Book of Physical Chemistry. By Dr. R. A. LEHFELDT, Professor of Physics at the East London Technical College. With 40 Illustrations. Crown 8vo., cloth, 7s. 6d.

Physical Chemistry for Beginners. By Dr. VAN DEVENTER. Translated by Dr. R. A. LEHFELDT. 2s. 6d.

The Standard Course of Elementary Chemistry. By E. J. COX, F.C.S., Headmaster of the Technical School, Birmingham. In Five Parts, issued separately, bound in cloth and illustrated. Parts I.-IV., 7d. each ; Part V., 1s. The complete work in one vol., 3s.

Practical Physics. A Preliminary Course of Practical Physics. By C. E. ASHFORD, M.A., Headmaster of the Royal Naval College, Osborne. Fcap. 4to., 1s. 6d.

Electrolytic Preparations. Exercises for use in the laboratory by Chemists and Electro-Chemists. By Dr. KARL ELBS, Professor of Organic and Physical Chemistry at the University of Giessen. Translated by R. S. HUTTON, M.Sc. Demy 8vo., 4s. 6d. net.

Electric and Magnetic Circuits. By E. H. CRAPPER, M.I.E.E., Head of the Electrical Engineering Department in University College, Sheffield. Demy 8vo., 10s. 6d. net.

Electricity and Magnetism. By C. E. ASHFORD, M.A., Headmaster of Osborne Royal Naval College, late Senior Science Master at Harrow School. With over 200 Diagrams. Cloth, 3s. 6d.

Magnetism and Electricity. By J. PALEY YORKE, of the Northern Polytechnic Institute, Holloway. Crown 8vo., cloth, 3s. 6d.

Physiology for Beginners. By LEONARD HILL, M.B. 1s.

Elementary Natural Philosophy. By ALFRED EARL, M.A., Senior Science Master at Tonbridge School. Crown 8vo., 4s. 6d.

A Text-Book of Zoology. By G. P. MUDGE, A.R.C.Sc. Lond., Lecturer on Biology at the London School of Medicine for Women. With about 150 Illustrations. Crown 8vo., 7s. 6d.

A Class-Book of Botany. By G. P. MUDGE, A.R.C.Sc., and A. J. MASLEN, F.L.S., Lecturer on Botany at the Woolwich Polytechnic. With over 200 Illustrations. Crown 8vo., 7s. 6d.

Psychology for Teachers. By C. LLOYD MORGAN, F.R.S., Principal of University College, Bristol. Crown 8vo., cloth, 3s. 6d.

LONDON : EDWARD ARNOLD, 41 & 43 MADDOX STREET, W

HISTORY.

A History of England. By C. W. OMAN, M.A., Deputy Professor of Modern History in the University of Oxford. Fully furnished with Maps, Plans of the Principal Battlefields, and Genealogical Tables. 760 pages. Ninth and Enlarged Edition (to end of South African War). Crown 8vo., cloth, 5s.

Special Editions, each volume containing a separate index.

In Two Parts, 3s. each: Part I., from the Earliest Times to 1603 ; Part II., from 1603 to 1902.

In Three Divisions : Division I., to 1307, 2s. ; Division II., 1307 to 1688, 2s. ; Division III., 1688 to 1902, 2s. 6d.

** *In ordering please state the period required, to avoid confusion.*

England in the Nineteenth Century. By C. W. OMAN, M.A., Author of "A History of England," etc. With Maps and Appendices. Revised and Enlarged Edition. One vol., crown 8vo., 3s. 6d.

A Junior History of England. From the Earliest Times to the Death of Queen Victoria. By C. W. OMAN, M.A., and MARY OMAN. With Maps. Cloth, 2s.

Questions on Oman's History of England. By R. H. BOOKEY, M.A. Crown 8vo., cloth, 1s.

A Synopsis of English History. By C. H. EASTWOOD, Headmaster of Redheugh Board School, Gateshead. 2s.

This useful little book is based upon Mr. Oman's "History of England," but can be used with any other text-book. It is designed upon an original plan, and will be found a valuable help in the study of history.

Seven Roman Statesmen. A detailed Study of the Gracchi, Cato, Marius, Sulla, Pompey, Cæsar. Illustrated with reproductions of Roman Coins from the British Museum. By C. W. OMAN. About 320 pages. Crown 8vo., cloth, 6s.
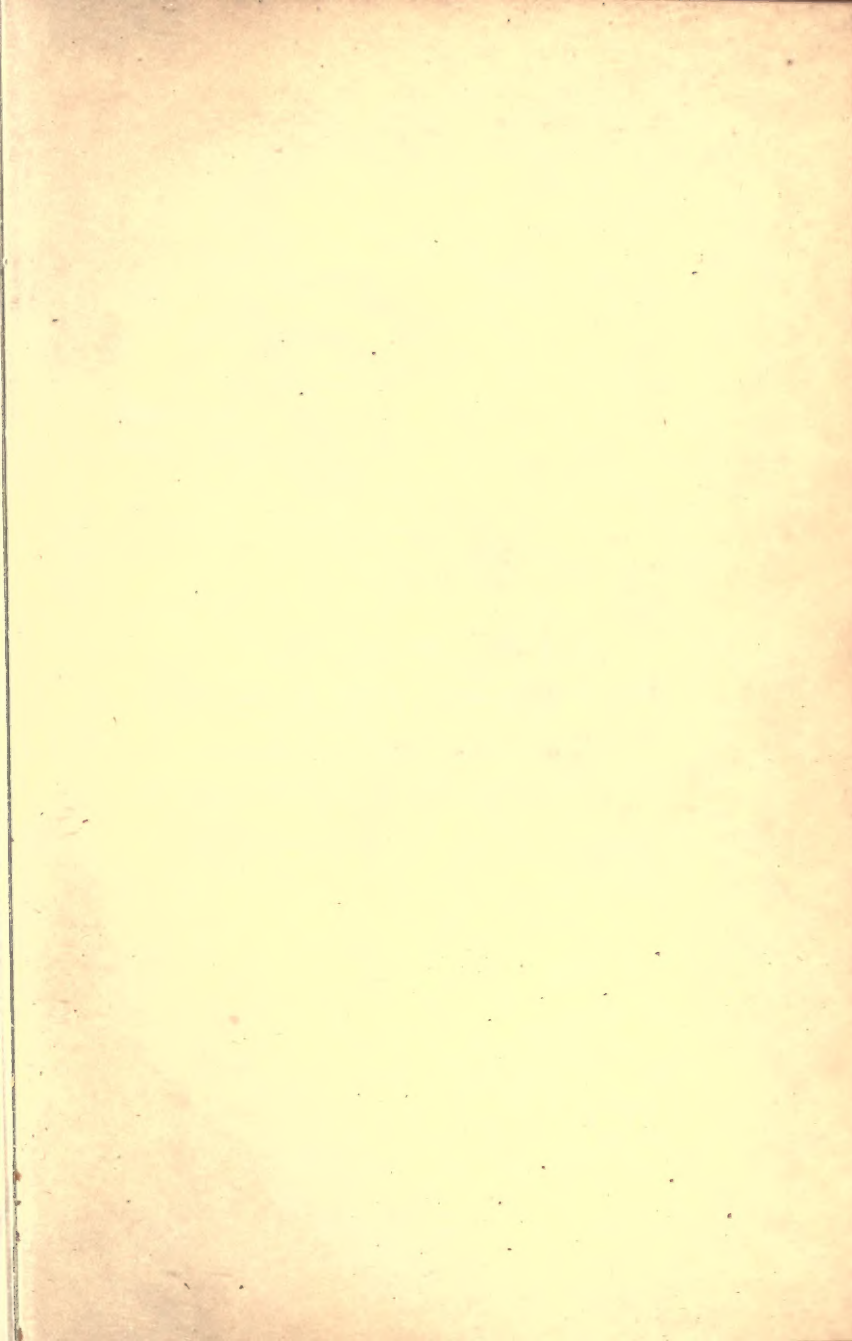
English History for Boys and Girls. By E. S. SYMES, Author of "The Story of Lancashire," "The Story of London," etc. With numerous Illustrations. One vol., 2s. 6d.
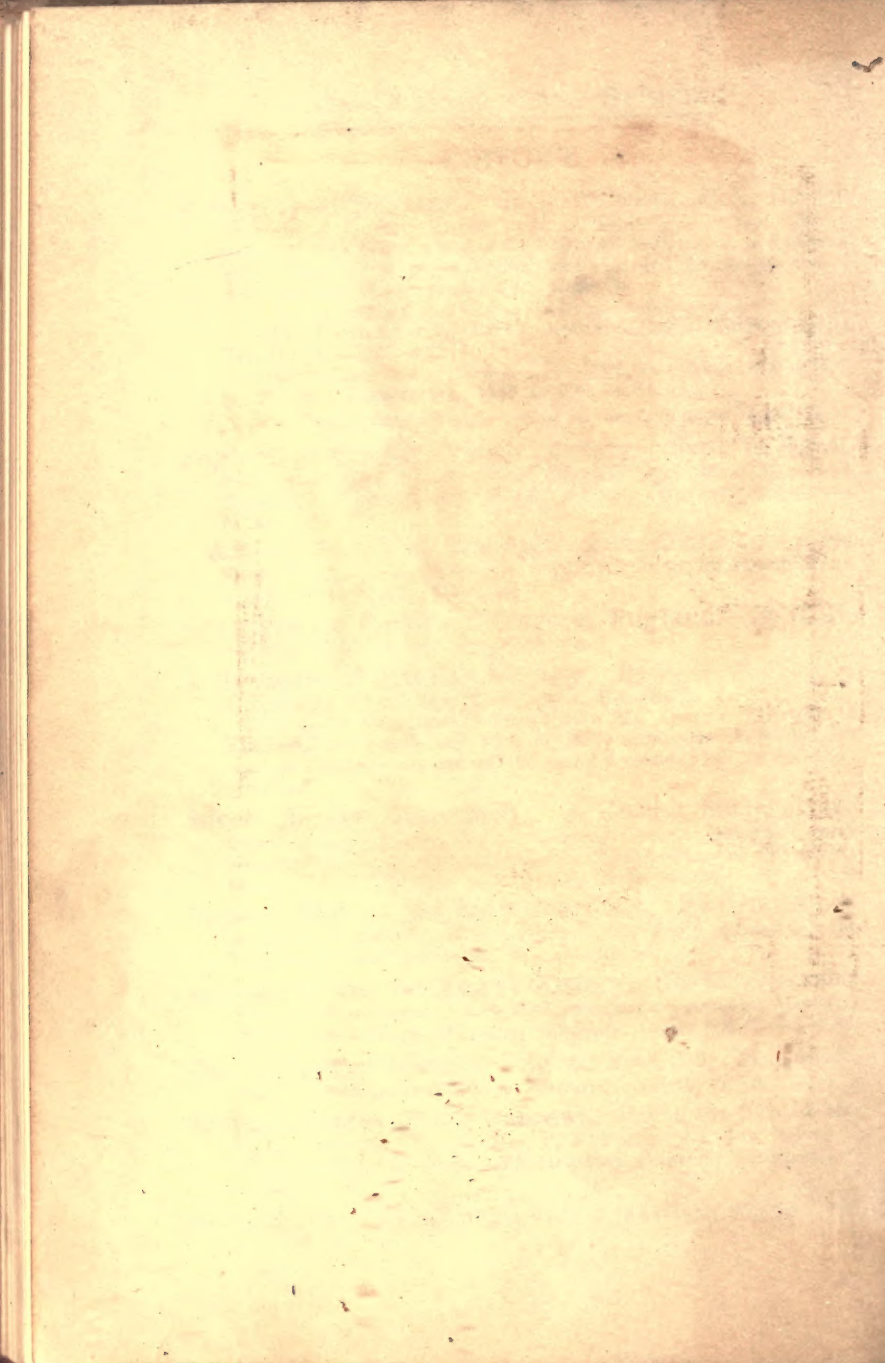
Lessons in Old Testament History. By the Venerable A. S. AGLEN, Archdeacon of St. Andrews, formerly Assistant Master at Marlborough College. 450 pages, with Maps. Crown 8vo., cloth, 4s. 6d.

Old Testament History. By the Rev. T. C. FRY, Headmaster of Berkhamsted School. Crown 8vo., cloth, 2s. 6d.

Biblical History of the Hebrews. Upon the lines of the Higher Criticism. By the Rev. F. J. FOAKES JACKSON, Fellow of Jesus College, Cambridge, and Honorary Canon of Peterborough. Crown 8vo., 6s.

LONDON : EDWARD ARNOLD, 41 & 43 MADDOX STREET, W.

12/7/04.